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Characterization of Continuous Selections of the Metric Projection for Spline Functions

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1. INTRODUCTION

Let $C = C[a, b]$ be the space of real-valued continuous functions on $[a, b]$ with the uniform norm. When G is an N-dimensional Chebyshev subspace of C, i.e., each f in C has a unique best approximation from G, then the mapping from f in C to its unique best approximation from G is continuous. When a space of functions in C is not a Chebyshev subspace the question arises whether one can select for each f in C a best approximation to f in such a way that this selection is continuous.

In this paper, we will characterize precisely when such a continuous selection exists. Indeed, we show that there is a continuous selection for spline functions if and only if the number of knots is smaller than or equal to the order of the splines.

If G is a nonempty set from C, let

$$P(f) = P_G(f) = \{g_0 \in G: \|f - g_0\| = \inf\{\|f - g\|: g \in G\}\}.$$

$P(f)$ is the set of best approximations to f from G. P is a set-valued mapping from C into 2^G called the *metric projection* onto G. A continuous mapping s from C into G is said to be a *continuous selection for the metric projection* P (or, more briefly, continuous selection) if $s(f) \in P(f)$, for each $f \in C$.

A starting point for the study of the existence of continuous selections is the weak Chebyshev subspaces of C. In this direction, we have shown [8] that if G is a weak Chebyshev space of dimension N with the property that each $g \in G$ has at most N distinct zeroes, then G admits a continuous selection. In fact, the selection is quite simply: we take for each $f \in C$ the alternation element (Definition 2.1) $g_f \in G$. The existence of an alternation element is guaranteed in case G is weak Chebyshev by Jones and Karlovitz [3]. We have shown [8] when each $g \in G$ has at most N distinct zeroes then g_f is unique

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for each f. In fact, the condition that each g has at most N distinct zeroes is necessary and sufficient that g_f be unique for all f.

The prototypes of weak Chebyshev spaces are the spline spaces $S_{n,k}$, where the splines are of degree n with k fixed knots. A spline $s \in S_{n,k}$ can vanish identically on an interval. Hence the results of [8] are not applicable.

In the case that functions from G have zero intervals, there is only one result in the literature guaranteeing the existence of continuous selections, and that is the theorem for one-dimensional subspaces in $C(X)$ given by Lazar *et al.* [7, Proposition 2.6].

The question of the existence of continuous selections for N-dimensional subspaces G ($N > 1$) with the property that functions in G may vanish on intervals has not been treated in the literature.

The techniques of [8] do not apply directly because, in general, splines admit more than one alternation element. However, it is possible by modifying the approach in [8] and using some well-known results from spline theory to construct continuous selections provided $k \leqslant n + 1$.

The construction of the selection is highly local and based on local alternation elements, whose uniqueness is guaranteed by the condition $k \leqslant n + 1$.

For $k > n + 1$, we show the nonexistence of continuous selections for $S_{n,k}$.

2. Preliminaries

In this section, we will summarize some of the known results about weak Chebyshev systems and splines that we will need to prove our characterization theorem. An important role in the construction of our selection is played by alternation elements.

Definition 2.1. Let G be an N-dimensional subspace of $C[a, b]$.

If $f \in C[a, b]$, then $g \in P(f)$ is called an *alternation element* (A element) for f if there exist $N + 1$ points $a \leqslant x_0 < x_1 < \cdots < x_N \leqslant b$ such that $\epsilon(-1)^i(f - g)(x_i) = \|f - g\|$, $i = 0, \ldots, N$, with $\epsilon = \pm 1$. The points $x_0, \ldots, x_N$ are called alternating extreme points of $f - g$.

Jones and Karlovitz [3] have characterized the N-dimensional subspaces G of $C[a, b]$ such that each f in $C[a, b]$ has at least one A element. The result is

Theorem 2.1. *G is weak Chebyshev if and only if for each $f \in C[a, b]$ there exists at least one* A *element in $P(f)$.*

Recall that a subspace G is called *weak Chebyshev* if each $g \in G$ has at most $N - 1$ changes of sign, i.e., there do not exist points $a \leqslant x_0 < \cdots < x_N \leqslant b$ such that $g(x_i) \cdot g(x_{i+1}) < 0$, $i = 0, \ldots, N - 1$.

We will establish in Section 3 a characterization when spline spaces have a continuous selection. This can be accomplished even in the general context of Chebyshevian splines. Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ be k fixed knots in $[a, b]$ and $\{w_i\}_{i=0}^n$ be $n + 1$ positive functions in $C[a, b]$ such that w_i is in $C^{n-i}[a, b]$, $i = 0,..., n$, $n \geqslant 1$. Then

$$
\begin{aligned}
u_0(x) &= w_0(x),\\
u_1(x) &= w_0(x) \int_a^x w_1(\xi_1)\, d\xi_1\,,\\
&\vdots\\
u_n(x) &= w_0(x) \int_a^x w_1(\xi_1) \int_a^{\xi_1} w_2(\xi_2) \cdots \int_a^{\xi_{n-1}} w_n(\xi_n)\, d\xi_n \cdots d\xi_1
\end{aligned}
$$

is said to be an *extended complete Chebyshev system.* We define

$$
\phi_n(x, t) := \begin{cases} w_0(x) \int_t^x w_1(\xi_1) \int_t^{\xi_2} w_2(\xi_2) \cdots \int_t^{\xi_{n-1}} w_n(\xi_n)\, d\xi_n \cdots d\xi_1\,, & x > t\\ 0, \quad x \leqslant t. \end{cases}
$$

The class of the *Chebyshevian spline functions* is defined by $S_{n,k}(x_1 ,..., x_k) := \langle u_0 ,..., u_n , \phi_n(\cdot, x_1),..., \phi_n(\cdot, x_k)\rangle$. The Chebyshevian splines form an $n + k + 1$-dimensional subspace of $C[a, b]$. In the following we denote this subspace simply by $S_{n,k}$. Each $g \in S_{n,k}$ is in $C^{n-1}[a, b]$ and the restriction of g to $[x_i , x_{i+1}]$, which we denote by $g\,|_{[x_i, x_{i+1}]}$, represents a generalized polynomial. If we define $w_i(x) := i + 1$ for each $x \in [a, b]$, $i = 0,..., n$, then $S_{n,k}$ will be the class of the usual *polynomial splines.* For the proof of our characterization we use some results from spline theory. First, we have the following characterization of best approximations given by Rice [9] and Schumaker [10].

THEOREM 2.2. *A function $g_0 \in S_{n,k}$ is a best approximation of $f \in C[a, b]$ if and only if $f - g_0$ has $n + j + 1$ alternating extreme points on some subinterval $[x_i , x_{i+j}]$.*

Rice [9] has in addition shown the existence of certain uniqueness intervals for spline approximation.

THEOREM 2.3. *Let $g_0 \in S_{n,k}$ be a best approximation of $f \in C[a, b]$ such that $f - g_0$ has $n + j + 1$ alternating extreme points in $[x_i , x_{i+j}]$, but does not have $n + l + 1$ alternating extreme points in any subinterval $[x_r , x_{r+l}]$ of $[x_i , x_{i+j}]$. Then all best approximations of f coincide on $[x_i , x_{i+j}]$.*

While the best approximation by splines is generally not unique, we have the following sufficient condition for uniqueness given by Strauss [12].

THEOREM 2.4. *Let $g_0 \in S_{n,k}$ be a best approximation of $f \in C[a, b]$ such that $f - g_0$ has at least $j + 1$ alternating extreme points in each interval*

(1) $[a, x_j), (x_{k-j+1}, b], j = 1,..., k,$

(2) $(x_i, x_{i+j+n}) \subset (a, b), j > 0$ $(k > n + 1)$.

Then g_0 is the unique best approximation of f.

The next theorem gives a restricted interpolation property for splines given by Karlin [4, p. 503].

THEOREM 2.5. *Let z_i, t_i, $i = 1,..., r$, be points such that*

$$a \leqslant z_1 < z_2 < \cdots < z_r \leqslant b,$$
$$a \leqslant t_1 < t_2 < \cdots < t_r \leqslant b;$$

then

$$\begin{vmatrix} \phi_n(z_1, t_1) & \cdots & \phi_n(z_1, t_r) \\ \vdots & & \vdots \\ \phi_n(z_r, t_1) & \cdots & \phi_n(z_r, t_r) \end{vmatrix} \geqslant 0.$$

Strict inequality holds when (1) $z_{i-n-1} < t_i < z_i$, $i = 1,..., r$. *For* $i \leqslant n + 1$, *the first inequality in* (1) *is omitted.*

Remark 2.1. From Karlin and Studden [6, p. 3], and Theorem 2.5 it follows that for arbitrary points t_i, $i = 1,..., r$, with

$$a \leqslant t_1 < t_2 < \cdots < t_r \leqslant b,$$

the functions $\phi_n(x, t_1),..., \phi_n(x, t_r)$ span an r-dimensional weak Chebyshev subspace of $C[a, b]$.

According to Schumaker [11] for each $g \in S_{n,k}$, which has only finitely many distinct zeroes in $[a, b]$, we count the zeroes in the following way (notice that we only consider simple knots):

DEFINITION 2.2. Let g be a function in $S_{n,k}$ which has only finitely many distinct zeroes in $[a, b]$. If z is a zero of $g \in S_{n,k}$ and z is not a knot, then z is said to be a *zero of multiplicity r* provided $g(z) = g'(z) = \cdots = g^{(r-1)}(z) = 0$, $g^{(r)}(z) \neq 0$. If z is a knot, then we use the same count because of $g \in C^{n-1}[a, b]$ for $r \leqslant n - 1$. If z is a knot and $g(z) = g'(z) = \cdots = g^{(n-1)}(z) = 0$, then z is said to be a *zero of multiplicity n* provided $g^{(n-1)}$ changes sign at z, and z is said to be a *zero of multiplicity n* + 1 provided $g^{(n-1)}$ does not change sign at z. We define $Z^*(g)$ to be the number of the zeroes of $g \in S_{n,k}$ counting multiplicities.

Schumaker [11] has estimated the number of zeroes a spline function in $S_{n,k}$ can possess. For our problem we need this result when $g \in S_{n,k}$ has only finitely many distinct zeroes.

THEOREM 2.6. *If $g \in S_{n,k}$ has only finitely many distinct zeroes then*

$$Z^*(g) \leqslant n + k.$$

We will prove a local uniqueness for A elements in $S_{n,k}$, using Theorem 2.7 [8] which counts the zeroes of a function $g_1 - g_2$ when g_1 and g_2 are both alternation elements.

THEOREM 2.7. *Let G be an m-dimensional weak Chebyshev subspace of $C[a, b]$ and $f \in C[a, b]$. If $g_1, g_2 \in P(f)$ are two alternation elements for $f \in C[a, b]$, then at least one of the following is true*:

(1) *$g_1 - g_2$ has at least $m + 1$ distinct zeroes in $[a, b]$,*

(2) *$g_1 - g_2$ has at least $m + 2$ zeroes in $[a, b]$.*

In condition (2) *we count zeroes in the following way*:

A zero z of $g_1 - g_2$ is said to be a simple zero *if $g_1 - g_2$ changes sign at z or if $z = a$ or $z = b$.*

A zero z in (a, b) of $g_1 - g_2$ is said to be a double zero *if $g_1 - g_2$ does not change sign at z.*

We are now in position to prove our Main Theorem.

3. CHARACTERIZATION THEOREM

THEOREM 3.1. *There exists a continuous selection s: $C[a, b] \to S_{n,k}$ for P: $C[a, b] \to 2^{S_{n,k}}$ if and only if $k \leqslant n + 1$.*

Proof.

Part 1. First we show the nonexistence of continuous selections in the case $k > n + 1$. This is done by constructing a function f and two sequences (f_m) and $(\tilde{f}_m)$ in $C[a, b]$ which converge to f with the following properties:

$$P_{S_{n,k}}(f_m) = \{0\} \qquad \text{and} \qquad P_{S_{n,k}}(\tilde{f}_m) = \{g_0\}, \quad g_0 \neq 0, \qquad \text{for each } m.$$

It is easy to see that this precludes the existence of a continuous selection s, since we would have $s(f) = 0$ because $f_m \to f$ and $s(f_m) = 0$, and $s(f) = g_0$ because $\tilde{f}_m \to f$ and $s(\tilde{f}_m) = g_0$. This would be a contradiction because $g_0 \neq 0$. Start with k fixed distinct knots $a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b$ and consider the interval $[x_1, x_{n+2}]$. By Karlin [4, p. 524], there exists a $\tilde{g}$ in $S_{n,k}$ such that

$$\tilde{g}(x) = 0 \qquad \text{if} \quad x \in [a, x_1] \cup [x_{n+2}, b]$$

and

$$\tilde{g}(x) \neq 0 \qquad \text{if} \quad x \in (x_1, x_{n+2}).$$

We may assume that

$$\tilde{g}(x) > 0 \qquad \text{if} \quad x \in (x_1, x_{n+2}).$$

Let $g_0 = \|\tilde{g}\|^{-1} \cdot \tilde{g}$. By Curry and Schoenberg [2, Theorem 1], g_0' has exactly one zero $\tilde{x} \in (x_1, x_{n+2})$ and hence g_0 has exactly one maximum at $\tilde{x}$. (In the literature this function is called a B-spline.)

Construction of f. We construct a function f with $\|f\| = 1$. First, define $f(x_1) = 1$, $f(x_{n+2}) = -1$, $f(\tilde{x}) = 0$. Next for each interval I of the form $[a, x_1]$, $[x_{n+2}, x_{n+3}]$,..., $[x_k, b]$, we require that f has exactly $n + 2$ alternating extreme points in I. We take x_1 as an alternating extreme point for $[a, x_1]$ and correspondingly x_{n+2} as one for $[x_{n+2}, x_{n+3}]$. Let $\{z_1, z_2, ..., z_\lambda\}$ be the set of all these extreme points. Then, $|f(z_i)| = 1$, $i = 1, 2, ..., \lambda$. Let f be linear on each $[z_i, z_{i+1}]$, except for $[x_1, x_{n+2}]$. Here we define f in the following way: Let f be linear in $[x_1, \tilde{x}]$ and $f(x) = g_0(x) - 1$ for $x \in [\tilde{x}, x_{n+2}]$. This is the construction of f. Let us note a couple of properties of f. By Theorem 2.2 and the position of the alternating extreme points of f it follows that $0, g_0 \in P_{S_{n,k}}(f)$. By Theorem 2.3, it even follows that all best approximations of f vanish identically in $[a, x_1] \cup [x_{n+2}, b]$. Note that in the case $k \leqslant n + 1$ there does not exist such an interval $[x_i, x_{i+n+1}] \subset [x_1, x_k]$. But by Theorem 2.5 for any interval $[x_i, x_{i+m+1}]$, $m < n$ there does not exist a g in $S_{n,k}$, $g \neq 0$, such that

$$g(x) = 0 \qquad \text{if} \quad x \in [a, x_i] \cup [x_{i+m+1}, b]$$

and

$$g(x) \neq 0 \qquad \text{for each } x \in (x_i, x_{i+m+1}).$$

Therefore the preceding method is not applicable when $k \leqslant n + 1$. Now let $\tilde{z} < x_1$ be the $n + 1$ alternating extreme point of f in $[a, x_1]$ and $\tilde{\tilde{z}} > x_{n+2}$ the second alternating extreme point of f in $[x_{n+2}, x_{n+3}]$.

Construction of the sequence $(\tilde{f}_m)$. We define for sufficiently large m

$$\begin{aligned}
\tilde{f}_m(x) &:= f(x), && x \in [a, \tilde{z}] \\
&= 1 + g_0\left(x_1 + \frac{1}{m}\right), && x = x_1 + \frac{1}{m} \\
&= 0, && x = \tilde{x} \\
&= -1 + g_0\left(x_{n+2} - \frac{1}{m}\right), && x = x_{n+2} - \frac{1}{m} \\
&= g_0(x) - 1, && x \in \left[\tilde{x}, x_{n+2} - \frac{1}{m}\right] \\
&= f(x), && x \in [\tilde{\tilde{z}}, b].
\end{aligned}$$

Let $\tilde{f}_m$ be linear in $[\tilde{z}, x_1 + (1/m)]$, $[x_1 + (1/m), \tilde{x}]$, $[x_{n+2} - (1/m), \tilde{\tilde{z}}]$.

Construction of the sequence $(\tilde{f}_m)$. We define for sufficiently large m

$$
\begin{aligned}
\tilde{f}_m(x) &= f(x), && x \in [a, \tilde{z}] \\
&= 1, && x = x_1 + \frac{1}{m} \\
&= 0, && x = \tilde{x} \\
&= \frac{g_0(x) - 1}{1 - g_0(x_{n+2} - (1/m))}, && x \in \left[\tilde{x}, x_{n+2} - \frac{1}{m}\right] \\
&= -1, && x = x_{n+2} - \frac{1}{m} \\
&= f(x), && x \in [\tilde{\tilde{z}}, b].
\end{aligned}
$$

Let $\tilde{f}_m$ be linear in $[\tilde{z}, x_1 + (1/m)]$, $[x_1 + (1/m), \tilde{x}]$, $[x_{n+2} - (1/m), \tilde{\tilde{z}}]$. This is the construction of the sequences (f_m) and $(\tilde{f}_m)$. Now, $(f_m), (\tilde{f}_m) \subset C[a, b]$ and $f_m \to f, \tilde{f}_m \to f$. From the fact that $\|f_m\| \leqslant 1$ and Theorem 2.2 we have 0 is in $P_{S_{n,k}}(f_m)$. Similarly $\|\tilde{f}_m - g_0\| \leqslant 1$ and so Theorem 2.2 gives that g_0 is in $P_{S_{n,k}}(\tilde{f}_m)$. Notice that g_0 is strictly monotone decreasing in $[\tilde{x}, x_{n+2}]$. By Theorem 2.4 it follows further that $P_{S_{n,k}}(f_m) = \{0\}$ and $P_{S_{n,k}}(\tilde{f}_m) = \{g_0\}$. This proves the nonexistence of a continuous selection in the case $k > n + 1$.

Part 2. We will now show the existence of a continuous selection in the case $k \leqslant n - 1$.

(1) *Construction of the selection* $s\colon C[a, b] \to S_{n,k}$. Let f be in $C[a, b]$, g_0 in $P_{S_{n,k}}(f)$, and $[x_l, x_{l+1}] \subset [a, b]$ be some interval on which all of the best approximations of f coincide. The existence of $[x_l, x_{l+1}]$ follows from Theorem 2.3.

(a) We first approximate $f - g_0$ in $[x_{l+1}, b]$ by elements from $G_{l+1} := \langle \phi_n(\cdot, x_{l+1}), \ldots, \phi_n(\cdot, x_k) \rangle \subset S_{n,k}$. According to Remark 2.1, G_{l+1} is a $(k - l)$-dimensional weak Chebyshev subspace of $C[a, b]$.

Therefore, Theorem 2.1 guarantees the existence of an A element g_1 in $P_{G_{l+1}}(f - g_0)$ for which

$$
\begin{aligned}
\|f - g_0 - g_1\|_{[a, x_{l+1}]} &= \|f - g_0\|_{[a, x_{l+1}]} \leqslant \|f - g_0\|, \\
\|f - g_0 - g_1\|_{[x_{l+1}, b]} &\leqslant \|f - g_0 - 0\|_{[x_{l+1}, b]} \leqslant \|f - g_0\|.
\end{aligned}
$$

So $g_0 + g_1$ is also in $P_{S_{n,k}}(f)$.

(b) We will now show for approximation in $[x_{l+1}, b]$, any two distinct A elements g_1, g_2 in $P_{G_{l+1}}(f - g_0)$ are the same on $[x_{l+1}, x_{l+2}]$, i.e., $g_1 = g_2$ in $[x_{l+1}, x_{l+2}]$. Assume to the contrary that $g_1 \neq g_2$ in $[x_{l+1}, x_{l+2}]$. We will show that $g_1 - g_2$ has no zero interval in $[x_{l+1}, b]$. If $g_1 - g_2$ has a zero interval then there exist $r - l$ points $t_1 < \cdots < t_{r-l}$

in some interval $[x_r, x_{r+l}] \subset [x_{l+2}, b]$ such that $(g_1 - g_2)(t_i) = 0$, $i = 1,..., r - l$. Since $g_1 - g_2$ is in G_{l+1}

$$g_1 - g_2 \,|_{[x_{l+1}, x_{r+1}]}\, (t_i) = \sum_{j=l+1}^{r} a_j \phi_n(t_i, x_j) = 0, \qquad i = 1,..., r - l.$$

By Theorem 2.5 it follows that $a_{l+1} = \cdots = a_r = 0$. This is a contradiction to $g_1 \neq g_2$ in $[x_{l+1}, x_{l+2}]$. Therefore $g_1 - g_2$ has no zero interval in $[x_{l+1}, b]$ and since $g_1 - g_2$ is in $S_{n,k}$ only finitely many zeroes. In particular $g_1 - g_2 \,|_{[x_{l+1}, b]}$ is in

$$S_{n,k-l-1}(x_{l+2},..., x_k) = \langle u_0,..., u_n, \phi_n(\cdot, x_{l+2}),..., \phi_n(\cdot, x_k)\rangle.$$

Therefore by Theorem 2.6 it follows $Z^*(g_1 - g_2) \leqslant n + k - l - 1$ in $[x_{l+1}, b]$. Notice that $g_1 - g_2$ has a zero of multiplicity n at x_{l+1} according to Definition 2.2. If we count the zeroes of $g_1 - g_2$ according to Theorem 2.7 $g_1 - g_2$ has at most $k - l$ zeroes in $[x_{l+1}, b]$. Since g_1 and g_2 are A elements in $P_{G_{l+1}}(f - g_0)$ it follows by Theorem 2.7 that $g_1 - g_2$ fulfills at least one of the following conditions: $g_1 - g_2$ has at least $k - l + 1$ distinct zeroes in $[x_{l+1}, b]$, $g_1 - g_2$ has at least $k - l + 2$ zeroes in $[x_{l+1}, b]$. This is a contradiction. Therefore $g_1 = g_2$ in $[x_{l+1}, x_{l+2}]$.

(c) We show: If $\tilde{g}_0 \in P_{S_{n,k}}(f)$, $\tilde{g}_0 \neq g_0$, and $\tilde{g}_1 \in P_{G_{l+1}}(f - \tilde{g}_0)$ is A element for approximation in $[x_{l+1}, b]$ then $g_0 + g_1 = \tilde{g}_0 + \tilde{g}_1$ in $[x_{l+1}, x_{l+2}]$. Since $g_0 = \tilde{g}_0$ in $[x_l, x_{l+1}]$, the function

$$\bar{g}_0(x) = \begin{cases} (g_0 - \tilde{g}_0)(x) & x \in [x_{l+1}, b], \\ 0 & x \in [a, x_{l+1}] \end{cases}$$

is in G_{l+1}.

The functions $f - g_0 - g_1$ and $f - \tilde{g}_0 - \tilde{g}_1 = f - g_0 - (-g_0 + \tilde{g}_0 + \tilde{g}_1)$ have $k - l + 1$ alternating extreme points in $[x_{l+1}, b]$. Since $\bar{g}_0$ is in G_{l+1} the function $\bar{g}_1 = \tilde{g}_1 - \bar{g}_0$ is also in G_{l+1} and this function is an A element of $f - g_0$ by approximation in $[x_{l+1}, b]$. Since according to (1b) all of these A elements coincide in $[x_{l+1}, x_{l+2}]$, we must have $g_1 = -g_0 + \tilde{g}_0 + \tilde{g}_1$ in $[x_{l+1}, x_{l+2}]$, as desired.

(d) This method will be continued in $[x_{l+2}, b]$ in the following way. We now approximate $f - g_0 - g_1$ in $[x_{l+2}, b]$ by $G_{l+2} = \langle \phi_n(\cdot, x_{l+2}),..., \phi_n(\cdot, x_k)\rangle$ and by Theorem 2.1 we get an A element g_2 in $P_{G_{l+2}}(f - g_0 - g_1)$. As in (1b), we have that all these A elements coincide in $[x_{l+2}, x_{l+3}]$ and as in (1c) we have that $g_0 + g_1 + g_2$ is independent of the choice of g_0 and $g_0 + g_1$ in $[x_{l+2}, x_{l+3}]$. Also, we have that $g_0 + g_1 + g_2$ is in $P_{S_{n,k}}(f)$. We continue this method up to the last interval $[x_k, b]$ and get a function $\hat{g} = g_0 + g_1 + \cdots + g_{k-l}$ in $P_{S_{n,k}}(f)$, which is independent of the choice of $g_0, g_0 + g_1,..., g_0 + g_1 + \cdots + g_{k-l-1}$.

(e) Using the same kind of argument as in (c) and (d), we get a function $\hat{g} = g_{-l} + g_{-l+1} + \cdots + g_0$ in $P_{S_{n,k}}(f)$ where for each i g_{-i} is an A element in $P_{G_{l+1-i}}(f - g_{-i+1} - \cdots - g_0)$ in $[a, x_{l+1-i}]$, with $\hat{G}_i = \langle \phi_n(x_1, \cdot),..., \phi_n(x_i, \cdot)\rangle$. As before $\hat{g}$ is independent of the choice of $g_0, g_{-1} + g_0,..., g_{-l+1} + \cdots + g_{-1} + g_0$. Define now $s(f) = g_{-l} + \cdots + g_{-1} + g_0 + g_1 + \cdots + g_{k-l}$ which is an element of $P_{S_{n,k}}(f)$.

If we have a function $g \in S_{n,k}$ which is zero on $I_1 = [z_1, z_2]$ and $I_2 = [z_3, z_4]$ with $z_2 < z_3$ then by Theorem 2.5 it follows that for $k \leqslant n + 1$ g also vanishes on $[z_2, z_3]$. Therefore $I_f := \{x \in [a, b]: g(x) = \tilde{g}(x)$ for each $g, \tilde{g} \in P_{S_{n,k}}(f)\}$ has to be an interval. Starting with an arbitrary interval $[x_l, x_{l+1}] \subset I_f$ therefore we get $s(f)$ independent of the choice of $[x_l, x_{l+1}] \subset I_f$.

(2) We show that s is continuous. Assume to the contrary that s is not continuous. Since $S_{n,k}$ is finite dimensional there exists a function f in $C[a, b]$ and a sequence (f_m) in $C[a, b]$ such that $f_m \to f$ and $s(f_m) \to g$, $g \neq s(f)$, $g \in P_{S_{n,k}}(f)$. Moreover, there are only a finite number of intervals possible for I_{f_m} (I_{f_m} is defined above). Hence, we can require that there is an interval I with $I_{f_m} = I$ for all m.

(a) We show first that there exists an interval $[x_p, x_{p+i}]$ which is contained in $I \cap I_f$.

By Theorems 2.2 and 2.3 there exists a subsequence of (f_m) which for notational convenience we again denote by (f_m) and an interval $[x_r, x_{r+j}] \subset I$ such that $f_m - s(f_m)$ has $n + j + 1$ alternating extreme points

$$x_r \leqslant z_0^{(m)} < \cdots < z_{n+j}^{(m)} \leqslant x_{r+j}$$

such that

$$\epsilon(-1)^i (f_m - s(f_m))(z_i^{(m)}) = \| f_m - s(f_m)\|, \qquad i = 0,..., n + j,$$

with $\epsilon = \pm 1$.

There exists a convergent subsequence of $(z_i^{(m)})$, $i = 0,..., n + j$ which we again denote by $(z_i^{(m)})$ such that $\lim_{m\to\infty} z_i^{(m)} = z_i$, $i = 0,..., n + j$.

It follows that

$$\begin{aligned}
\epsilon(-1)^i \| f - g \| &= \epsilon(-1)^i \lim_{m\to\infty} \| f_m - s(f_m)\| \\
&= \lim_{m\to\infty} (f_m(z_i^{(m)}) - f(z_i^{(m)}) + f(z_i^{(m)}) \\
&\qquad - g(z_i^{(m)}) + g(z_i^{(m)}) - s(f_m)(z_i^{(m)})) \\
&= \lim_{m\to\infty} (f_m(z_i^{(m)}) - f(z_i^{(m)})) \\
&\qquad + \lim_{m\to\infty} (f(z_i^{(m)}) - g(z_i^{(m)})) \\
&\qquad + \lim_{m\to\infty} (g(z_i^{(m)}) - s(f_m)(z_i^{(m)})) \\
&= (f(z_i) - g(z_i)), \qquad i = 0,..., n + j.
\end{aligned}$$

Therefore $f - g$ has $n + j + 1$ alternating extreme points in $[x_r, x_{r+j}]$. Thus, by Theorem 2.3 there exists an interval $[x_p, x_{p+i}] \subset [x_r, x_{r+j}]$ on which all best approximations of f coincide. The interval $[x_p, x_{p+i}]$ is contained in $I \cap I_f$.

(b) According to (2a) there exists an interval $[x_l, x_{l+1}] \subset I_f \cap I$. Since s is independent of the choice of such an interval, $s(f)$ can also be defined by starting with $[x_l, x_{l+1}]$. Therefore, $s(f) = g_{-l} + \cdots + g_0 + \cdots + g_{k-l}$ and $s(f_m) = g_{-l,m} + \cdots + g_{0,m} + \cdots + g_{k-l,m}$. By choosing again a subsequence if necessary we can require that there is for each i, a $\tilde{g}_i$ with $\tilde{g}_i = \lim_{m\to\infty} g_{i,m}$ and of course $g = \tilde{g}_{-l} + \cdots + \tilde{g}_0 + \cdots + \tilde{g}_{k-l}$ with $\tilde{g}_i \in P_{G_{l+i}}(f - \tilde{g}_0 - \cdots - \tilde{g}_{i-1})$ for $i = 1, ..., k - l$ and $\tilde{g}_{-i} \in P_{G_{l+1-i}} \times (f - \tilde{g}_{-i+1} - \cdots - \tilde{g}_0)$ for $i = +1, ..., +1$.

In the construction of $s(f)$, there is some freedom in the selection of the functions g_i. For example, g_0 is a best approximation to f from $S_{n,k}$, etc. We will now show that for each i g_i can be taken as $\tilde{g}_i$. This is done by induction. We consider the case $i \geqslant 0$. The case $i < 0$ is proved similarly.

For $i = 0$, we have each $g_{0,m}$ is a best approximation to f_m. Since (f_m) converges to f $\tilde{g}_0 = \lim_{m\to\infty} g_{0,m}$ is also a best approximation to f. Hence, we can take $g_0 = \tilde{g}_0$ in the definition of $s(f)$.

Now suppose we know we can take $g_j = \tilde{g}_j$ for $0 \leqslant j < i$ with $i > 0$.

For each m $g_{i,m}$ is an A element to $f_m - (g_{0,m} + \cdots + g_{i-1,m})$ from G_{l+i} in $[x_{l+i}, b]$. Hence taking limits $\tilde{g}_i$ is an A element to $f - (\tilde{g}_0 + \cdots + \tilde{g}_{i-1})$ in $[x_{l+i}, b]$. Now it follows from (1b), (1c), (1d), and (1e) that $\tilde{g}_0 + \cdots + \tilde{g}_i = g_0 + \cdots + g_i$ in $[x_{l+i}, x_{l+i+1}]$ and thus we can take $g_i = \tilde{g}_i$ as desired by induction hypothesis.

Now that we have shown that for each i g_i can be taken as $\tilde{g}_i$, we have that $s(f) = g$. This is the desired contradiction and establishes the continuity of s. This completes the proof of the Theorem.

Since the results used in the proof of this theorem are also valid for knots with multiplicity less than $n + 1$ the characterization theorem is also true in this case.

For those weak Chebyshev subspaces of $C[a, b]$ such that each f in $C[a, b]$ admits exactly one A element g_f in $P(f)$ we can define the continuous selection s by $s(f) = g_f$ (see [8]).

It follows, however, by a result in [8] that a spline function in $S_{n,k}$ generally admits more than one A element. Here it would be natural to choose for each f in $C[a, b]$ an A element g_f in $P_{S_{n,k}}(f)$ with certain characteristic properties and to show the continuity of this selection. This, however, was not possible.

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